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Indian Methods of finding the approximate value of π and the development of calculus

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Abstract

Many Indian mathematicians calculate the approximate value of π. The value of π stated by Āryabhata – I was accepted by all the mathematicians which is (π = 22/7). Indian mathematicians used different methods to find the values of π. Madhava of Sangamgram of Kerala calculated the value of π in terms of infinite series. This method of finding the value of π is the beginning of idea of calculus in India. These methods are found in the text Yuktibhasha of Jyesthadeva (1500-1610 ad.),Tantrasangraha of Nilakantha (1443-1560 ad.),Kriyakramakari of Sankara Variyar (1500-1560 ad.). We have discussed here three methods of finding the value of π and also calculus involved in it. These three methods cover the idea of infinitesimal calculus. These methods are as follows.

Keywords:

Yukti-bhāṣā;
Kriyā-kramakarī;
Infinite series for π;
Regular polygon;
Properties of similar triangles.

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1. Introduction (10pt)

In India mathematics is developed for the astronomy. In the text of ancient Indian mathematicians are on astronomy which contains two to three chapters of mathematics necessary for the astronomical calculations. The value of π is necessary for the astronomical calculations. All Indian mathematicians knows the importance of most accurate value of π (value of π is approximate).

Indian mathematician and astronomer Āryabhata calculated the value of π by inscribing a regular hexagon in a circle. He knew the fact that the length of a side of such hexagon is equal to radius of a circle. Then doubling the sides of a inscribed hexagon we get a polygon of side 12 and continuing this process we obtain approximate value of π.

To obtain the value of correct up 11 decimal places Mādhava derived infinite series for π/4. This series for π/4 is slowly converging. It is so slow that for obtaining value of π correct to 2 decimal places we have to find hundreds of terms and for getting correct to 4 to 5 decimal places we have to consider million of terms. Mādhava knows the fact of end correction used in infinite series. Using the end correction he has given accurate value of π correct to 11 decimal places. The method used by Mādhava is known as *antya-*

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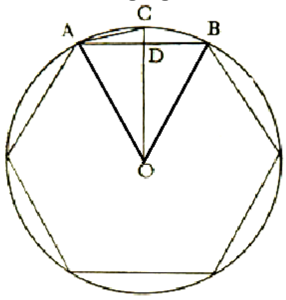
saṃskāra. The name itself indicates that the series is terminated after considering only some number of terms from the beginning.

These methods of finding the value of π cover the idea of infinitesimal calculus. We have discussed the infinitesimal calculus involved in finding the value of π in the text of Indian mathematicians

2. Inscribing a regular polygon in the circle.

Indian mathematician Āryabhata know the fact that a regular hexagon inscribed in a circle has side equal to the radius of circle. A regular polygon of side 12 can be inscribed in a circle by drawing the perpendicular on each side of a regular hexagon by joining the vertices of hexagon with the points of intersection of perpendicular on the side and the circle. We can calculate the length of side of a polygon of side 12. Continue the process and double the side of regular polygon and calculate the length of side of polygon. We can calculate the perimeter of polygon. The first ratio of perimeter of regular polygon and the diameter is 3. So the first approximate value of π is 3. This ratio of circumference and diameter increases every time but less than actual ratio of circumference and diameter. We can obtain the approximate value of π by this method. Following figure and table shows the clear idea of finding the value of π.



The general formula for calculating the side of a inscribed regular polygon of side 2n is

$$S_{2n} = \sqrt{\left(\frac{S_n^2}{4}\right) + \left(r - \sqrt{\frac{4r^2 - S_n^2}{4}}\right)^2}$$

Inscribed polygon	No. of sides	Length of side Put r = 1	Perimeter of polygon	Value of π
Hexagon	6	S_6 = 1	6	3
Polygon of sides 12	12	S_{12} = 0.5176	6.21165	3.10582
Polygon of sides 24	24	S_{24} = 0.261032	6.26478806	3.13239
Polygon of sides 48	48	S_{48} = 0.130806	6.278688	3.13934
Polygon of sides 96	96	S_{96} = 0.065438	6.282048	3.14102
Polygon of sides 192	192	S_{192} = 0.032723	6.28288896	3.14144
Polygon of sides 384	384	S_{384} = 0.016362	6.28309248	3.14154
Polygon of sides 768	768	S_{768} = 0.00818117	6.28314	3.14157 App. 3.1416

As the side of regular polygon becomes small we obtain a better approximation. This the approach of calculus to obtain the regular polygon of large number of sides and the side of regular polygon becomes smaller and smaller.

3 Circumference of a circle using circumscribing regular polygons

Suppose a circle of radius r = OB which is circumscribed by a square. In the figure a quadrant of square OAX_1B is circumscribing a quadrant of circle. OX_1 is the diagonal or hypotenuse (*karṇa*) joining the centre O and opposite corner X_1. Diagonal OX_1 meets the circle at point Z_1. AB is the other diagonal perpendicular to the Diagonal OX_1 at C_1. Denote half side of circumscribing square by S_1 (further we denote half side of regular octagon by S_2 and so on) and half the diagonal of regular polygon by K_1 (further we denote half the diagonal of regular octagon by K_2 and so on)

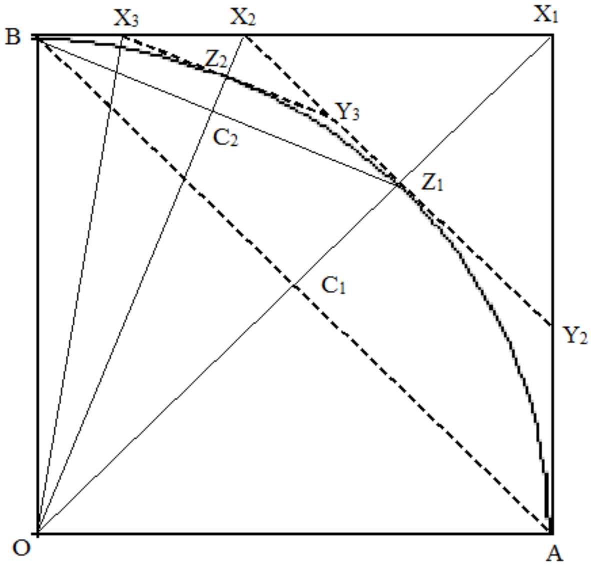
$S_1 = OB = BX_1 = r$ = sides of square = OAX_1B

$K_1 = OX_1 = \sqrt{2}\, r$

(Circumscribed polygon is a Square, no of sides 4, half length of side 1 = r, perimeter 8, diameter = 2, first value of π = 4).

$OC_1 = A_1 = X_1C_1 = \frac{OX1}{2} = \frac{\sqrt{2}\,r}{2} = \frac{r}{\sqrt{2}}$ (the triangle OC_1B and BC_1X_1 are isosceles right angled triangle with hypotenuse r).

$X_1C_1 = OC_1 = A_1$ is called base segment. At Z_1 draw a tangent to the circle, extend this tangent to meet BX_1 at X_2. BA and X_2Z_1 are parallel to each other. Triangles BC_1X_1 and $X_2Z_1X_1$ are similar triangles. Using rule of three



$\frac{X_1X_2}{X_1Z_1} = \frac{X_1B}{X_1C_1}$, $X_1X_2 = (K_1 - r)\left(\frac{S_1}{\frac{r}{\sqrt{2}}}\right)$

$BX_2 = S_2 = X_1B - X_1X_2 = S_1 - (K_1 - r)\left(\frac{S_1}{\frac{r}{\sqrt{2}}}\right)$

X_2Y_2 will result the side of a regular octagon, whose half side is BX_2 or X_2Z_1. Assuming the radius r = 1, half side of regular octagon is

$S_2 = BX_2 = 1 - (\sqrt{2} - 1)\left(\frac{1}{\frac{1}{\sqrt{2}}}\right) = 1 - (2 - \sqrt{2}) = 0.414$

Side of regular octagon of is 2 times 0.414 which is 0.828.

(In circumscribed octagon, no of sides 8, half length of side S_2 = 0.414, side is 0.828, perimeter of octagon 6.624, diameter of circle = 2, approximate value of π = 3.312).

Now join OX_2, which is half the diagonal of regular octagon, denote by K_2, now in triangle OBX_2

$OX_2 = K_2 = \sqrt{r^2 + S_2^2}$

Now we calculate the smaller base segment X_2C_2. Triangles OBX_2 and OZ_1X_2 are congruent right angled triangles, join B and Z_1, C_2 is point of intersection of OX_2 and BZ_1. BC_2 is perpendicular to OX_2. Using properties of right angled triangle for the triangle OBC_2 and BC_2X_2, we have

$(BX_2)^2 - (X_2C_2)^2 = (OB)^2 - (OC_2)^2 = (BC_2)^2$

or $(OC_2)^2 - (X_2C_2)^2 = (OB)^2 - (BX_2)^2$

Now $OC_2 + C_2X_2 = OX_2 = K_2$

$OC_2 - C_2X_2 = \frac{(OC_2)^2-(X_2C_2)^2}{OC_2+C_2X_2} = \frac{(OB)^2-(BX_2)^2}{OX_2} = \frac{(r)^2 - S_2^2}{K_2}$

$OX_2 - C_2X_2 - C_2X_2 = K_2 - 2\,C_2X_2 = \frac{(r)^2 - S_2^2}{K_2}$

$A_2 = C_2X_2 = \frac{\left(K_2 - \frac{((r)^2 - S_2^2)}{K_2}\right)}{2}$

Now draw Z_2X_3 such that it is parallel to BC_2, naturally X_3 is the point on BX_2. Triangles $X_2Z_2X_3$ and X_2C_2B are similar triangles using rule of three we have

$X_2X_3 = (X_2Z_2)\left(\frac{BX_2}{C_2X_2}\right) = (K_2 - r)\left(\frac{S_2}{A_2}\right)$

Hence half side of sixteen sided regular polygon is

$S_3 = BX_3 = BX_2 - X_2X_3 = S_2 - (K_2 - r)\left(\frac{S_2}{A_2}\right)$

(In circumscribed polygon of sides 16, r =1, K_2 = 1.0823, A_2 = 0.158352, X_2X_3 = 0.2152, half length of side S_3 = 0.1988, side is 0.3976, perimeter of sixteen sided polygon is 6.3616, diameter of circle = 2, approximate value of π = 3.1808).
Following the same sequence we can calculate the side of a regular polygon of sides 32.
We can generalise this result. If S_n is the half side of regular polygon of sides 2^n then the half side of regular polygon of sides 2^{n+1} is S_{n+1} calculated as follows

$K_n = \sqrt{r^2 + S_n^2}$, ...(1)

$A_n = \frac{\left(K_n - \frac{((r)^2 - S_n^2)}{K_n}\right)}{2}$, $A_n = \frac{S_n^2}{K_n}$ .....(2)

$S_{n+1} = S_n - (K_n - r)\left(\frac{S_n}{A_n}\right)$ ....(3)

Put 1 and 2 in 3 we have

$S_{n+1} = \frac{r}{S_n}\left(\sqrt{r^2 + S_n^2} - r\right)$

As the number of sides of regular polygon increases or no of corners of polygon becomes infinite we obtain the circumference of a circle.
This method was discussed in *Yukti-bhāṣā* and *Kriyā-kramakarī*.
In both the methods discussed above, we have to find a square root of number. The calculation becomes lengthy as the number of sides of a regular polygon increases.

4. Mādhava's Series for π

Now we discuss the method of calculating circumference without finding **square roots** is a contribution of Mādhava (1340- 1425 A.D.), based on concept of infinite series.

The infinite series is credited to Mādhava but quoted by Śaṅkara Vāriyar (1500 – 1560 A.D) in his commentary *Yukti-dīpikā* and *Kriyā-kramakarī*. The method of calculating circumference without finding square roots is the greatest contribution of Mādhava based on concept of infinite series. Also it covers the concept of Integration. This result gives us values of π. The quoted verse is as follows

O;kls okfjfèkfugrs :iârs O;kllkxjkfÒgrsA
f='kjkfnfo"kela[;kHkDra _.ka Loa i`FkDØekr~ dq;kZrAA

Meaning:- The diameter multiplied by four and divided by unity, decreases and increases should be made in turn of diameter multiplied by four and divided one by one by the odd numbers beginning with 3 and 5.
This series is written as

$$\text{Circumference} = 4d - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + \cdots$$

Or $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

The proof of this result is depends on the properties of similar triangles and several techniques including the ideas of integration and differentiation.
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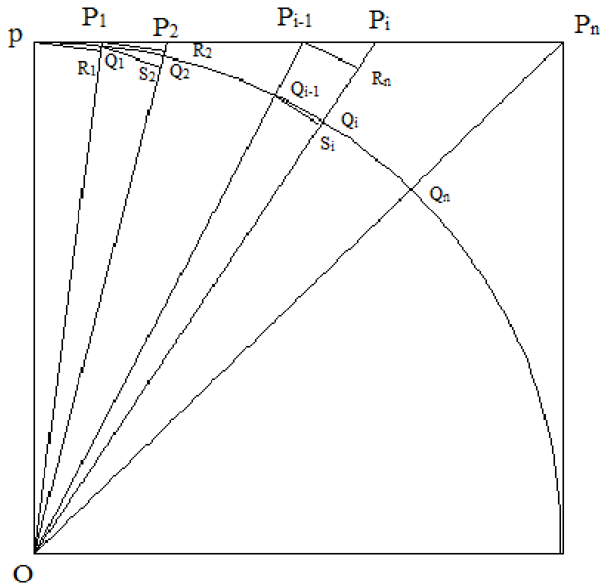

The circle is inscribed in a square of side equal to the diameter of the circle. This circle touches the middle points of the sides of square. A quarter of the circle with the circumscribing square is shown in the figure. The half side of square PP_n is divided into small number of equal parts PP_1, P_1P_2, P_2P_3, ... $P_{n-1}P_n$

each of length Δr. PP_1, PP_2, ... are called *bhujās.* The points P, P_1, P_2,.........P_n are joined to centre O. The line OP_1 cuts the circumference at Q_1, similarly line OP_2 cuts the circumference at Q_2,......and OP_n cuts the circumference at Q_n.

Lines OP_1, OP_2, ... OP_n are called as *karṇas*. From P, P_1, P_2,....perpendiculars PR_1, P_1R_2, ... are drawn on next *karṇas*. Means PR_1 is perpendicular from P on OP_1, P_1R_2 is perpendicular from P_1 on OP_2 similarly $P_{i-1}R_n$ is perpendicular from P_{i-1} on OP_i. Now from Q_1, Q_2,.. which are the points of intersection of *karṇas* and circumference of circle perpendiculars are drawn on next *karṇas*. Q_1S_2 is perpendicular on OP_2, Q_2S_3 is perpendicular on OP_3 and $Q_{i-1}S_i$ is perpendicular on OP_i.

Then from similar triangles OPP_1 and OPR_1

$$\frac{PR_1}{PP_1} = \frac{OP}{OP_1} \quad \text{or} \quad PR_1 = \frac{OP}{OP_1}PP_1 = \frac{\Delta r.r}{OP_1}$$

(OP = r is the radius of circle and Δr is the length each subdivision of PP_n). Again from similar triangles $P_1R_2P_2$ and POP_2

$$\frac{P_1R_2}{P_1P_2} = \frac{OP}{OP_2} \quad \text{OR} \quad P_1R_2 = \frac{OP\,.\,P_1P_2}{OP_2} = \frac{\Delta r.r}{OP_2}$$

Similarly

$$P_2R_3 = \frac{\Delta r.r}{OP_3},\ P_3R_4 = \frac{\Delta r.r}{OP_4} \quad \text{and} \quad P_{n-1}R_n = \frac{\Delta r.r}{OP_n}$$

Similarly triangles OP_1R_2 and OQ_1S_2 are similar

$$\frac{Q_1S_2}{P_1R_2} = \frac{OQ_1}{OP_1} \quad \text{OR} \quad Q_1S_2 = \frac{P_1R_2\,.\,OQ_1}{OP_1}$$

$$Q_1S_2 = \frac{\Delta r.r}{OP_2}\frac{r}{OP_1} = \frac{\Delta r.r^2}{OP_1.OP_2}$$

(Putting P_1R_2 from above result and OQ_1 is radius of circle r).

Similarly $Q_2S_3 = \frac{\Delta r.r}{OP_2}\frac{r}{OP_1} = \frac{\Delta r.r^2}{OP_2.OP_3}$

$$Q_{n-1}S_n = \frac{\Delta r.r^2}{OP_{n-1}.OP_n}$$

If the arcs are sufficiently small, or divisions of PP_n are sufficiently large then

$PR_1 = PQ_1$, $Q_1S_2 = Q_1Q_2$, ... $Q_{n-1}S_n = Q_{n-1}Q_n$

Hence

$$\text{arc } PQ_n = \frac{1}{8} \text{ of the circumference} = \frac{C}{8}$$

arc $PQ_n = PR_1 + Q_1S_2 + Q_2S_3 + ...+ Q_{n-1}S_n$

$$= \frac{\Delta r.r}{OP_1} + \frac{\Delta r.r^2}{OP_1.OP_2} + \frac{\Delta r.r^2}{OP_2.OP_3} + ...+ \frac{\Delta r.r^2}{OP_{n-1}.OP_n}$$

$$= \frac{\Delta r.r^2}{OP\,.\,OP_1} + \frac{\Delta r.r^2}{OP_1.OP_2} + \frac{\Delta r.r^2}{OP_2.OP_3} + ...+ \frac{\Delta r.r^2}{OP_{n-1}.OP_n} \quad (OP = r)$$

$$\frac{C}{8} \approx \Delta r\left(\left(\frac{r^2}{OP\,.\,OP_1}\right) + \left(\frac{r^2}{OP_1.OP_2}\right) + \left(\frac{r^2}{OP_2.OP_3}\right) + \ldots + \left(\frac{r^2}{OP_{n-1}.OP_n}\right)\right)$$

Now we can approximate it to any one of following expression

$$\left(\frac{C}{8}\right)_{left} = \Delta r\left(\left(\frac{r^2}{(OP_1)^2}\right) + \left(\frac{r^2}{(OP_2)^2}\right) + \left(\frac{r^2}{(OP_3)^2}\right) + \ldots + \left(\frac{r^2}{(OP_n)^2}\right)\right)$$

or

$$\left(\frac{C}{8}\right)_{right} = \Delta r\left(\left(\frac{r^2}{(OP)^2}\right) + \left(\frac{r^2}{(OP_1)^2}\right) + \left(\frac{r^2}{(OP_2)^2}\right) + \ldots + \left(\frac{r^2}{(OP_{n-1})^2}\right)\right)$$

The relation in above equations is

$$\left(\frac{C}{8}\right)_{left} < \frac{C}{8} < \left(\frac{C}{8}\right)_{right}$$

The actual value of circumference lies between $\left(\frac{C}{8}\right)_{left}$ and $\left(\frac{C}{8}\right)_{right}$

The difference of $\left(\frac{C}{8}\right)_{left}$ and $\left(\frac{C}{8}\right)_{right}$ is

$$\left(\frac{C}{8}\right)_{right} - \left(\frac{C}{8}\right)_{left} = \Delta r\left(\left(\frac{r^2}{(OP)^2}\right) - \left(\frac{r^2}{(OP_n)^2}\right)\right)$$

$$= \Delta r\left(\left(\frac{r^2}{r^2}\right) - \left(\frac{r^2}{2r^2}\right)\right) \text{ because op = r and op}_n = \sqrt{2}r$$

$$= \Delta r\left(1 - \left(\frac{1}{2}\right)\right) = \frac{\Delta r}{2}$$

As the number of partitions of PP_n increases length of each sub interval Δr approaches to zero. We have

$$\frac{C}{8} = \Delta r\left(\left(\frac{r^2}{(OP_1)^2}\right) + \left(\frac{r^2}{(OP_2)^2}\right) + \left(\frac{r^2}{(OP_3)^2}\right) + \ldots + \left(\frac{r^2}{(OP_n)^2}\right)\right)$$

Using binomial expansion for each term

$$\frac{\Delta r\, r^2}{(OP_i)^2} = \Delta r - \Delta r\left(\frac{(OP_i)^2 - r^2}{r^2}\right) + \Delta r\left(\frac{(OP_i)^2 - r^2}{r^2}\right)^2 - \ldots\ldots..$$

For the first term $(OP_1)^2 - r^2 = (\Delta r)^2$

$$\frac{\Delta r\, r^2}{(OP_1)^2} = \Delta r - \Delta r\frac{(\Delta r)^2}{r^2} + \Delta r\frac{(\Delta r)^4}{r^4} - \cdots = \Delta r(1 - \frac{(\Delta r)^2}{r^2} + \frac{(\Delta r)^4}{r^4} - \cdots).$$

For the second term $(OP_2)^2 - r^2 = (2\Delta r)^2$

$$\frac{\Delta r\, r^2}{(OP_2)^2} = \Delta r - \Delta r\frac{(2\Delta r)^2}{r^2} + \Delta r\frac{(2\Delta r)^4}{r^4} - \cdots..$$

For the last term $(OP_n)^2 - r^2 = (n\Delta r)^2$

$$\frac{\Delta r\, r^2}{(OP_n)^2} = \Delta r - \Delta r\frac{(n\Delta r)^2}{r^2} + \Delta r\frac{(n\Delta r)^4}{r^4} - \cdots..$$

We have

$$\frac{C}{8} = \Delta r\left(1 - \frac{(\Delta r)^2}{r^2} + \frac{(\Delta r)^4}{r^4} - \cdots\right)$$
$$+ \Delta r\left(1 - \frac{(2\Delta r)^2}{r^2} + \frac{(2\Delta r)^4}{r^4} - \cdots\right)$$
$$+ \Delta r\left(1 - \frac{(3\Delta r)^2}{r^2} + \frac{(3\Delta r)^4}{r^4} - \cdots\right) + \ldots\ldots..$$
$$\ldots\ldots.+ \Delta r\left(1 - \frac{(n\Delta r)^2}{r^2} + \frac{(n\Delta r)^4}{r^4} - \cdots\right)$$

$$\frac{C}{8} = n\Delta r - \Delta r\frac{(\Delta r)^2}{r^2}(1^2 + 2^2 + 3^2 + \cdots + n^2)$$
$$+ \Delta r\frac{(\Delta r)^4}{r^4}(1^4 + 2^4 + 3^4 + \cdots + n^4) + \ldots\ldots$$

Using sama-ghāta-saṅkalita $S_n^{(k)} \approx \frac{n^{k+1}}{k+1}$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n^3}{3},\ 1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{n^5}{5}$$

Using above results

$$\frac{C}{8} = n\Delta r - \Delta r\frac{(\Delta r)^2}{r^2}\frac{n^3}{3} + \Delta r\frac{(\Delta r)^4}{r^4}\frac{n^5}{5} - \cdots$$

Now put $n\Delta r = r$

$$\frac{C}{8} = r - \frac{r}{3} + \frac{r}{5} - \frac{r}{7} + \cdots$$

$$\frac{c}{8r} = \frac{c}{4d} = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$ as required.

This series does not converge rapidly. It is so slow that even for obtaining the value of π correct to two decimal places we have to find hundreds of terms and for getting π correct to 4 or 5 decimal places we have to find millions of terms.

To find the accurate value circumference we have to derive rapidly converging series. *Yukti-bhāṣa* deals with the rapidly converging series and contains many series derived from original series by grouping the elements. The new series obtained converges rapidly.

$$C = 4d\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$
$$= 4d\left(\left(1 - \frac{1}{3}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{9} - \frac{1}{11}\right) + \cdots\right)$$
$$= 8d\left(\left(\frac{1}{2^2-1}\right) + \left(\frac{1}{6^2-1}\right) + \left(\frac{1}{10^2-1}\right) + \cdots\ldots.\right)$$

Similarly

$$C = 4d - 4d\left(\left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{7} - \frac{1}{9}\right) + \cdots\ldots\right)$$
$$= 4d - 8d\left(\left(\frac{1}{4^2-1}\right) + \left(\frac{1}{8^2-1}\right) + \cdots\right)$$

5. Following important ideas of calculus are found in the development of series for π/4.

1) The development of series for π/4 is an important result. Without an best approximate value for π, sine tables for various angles cannot be constructed. This importance of value of π was well known to Āryabhata. The verses in Āryabhatīya, in which the value of π is stated as 3.1416, precede the two verses describing the preparation of the sine table.

2) Local linearization is a geometrical operation, but there is no unique way. *Yukti-bhāṣa* approach to linearising θ as a function of tan θ may be original and apparently different. This contains essential geometric input which is the orthogonality of radius and tangent at any point on the circle.

3) As the limit n → ∞ is taken at the end of all computations, the neglected quantities must be explicitly demonstrated to vanish when added up in the limit. *Yukti-bhāṣa* does this with care. Once that is done, on each calculation *sama-ghāta-saṅkalita* is, what is known now as Riemann integral.

This series for π/4 is slowly converging. It is so slow that for obtaining value of π correct to 2 decimal places we have to find hundreds of terms and for getting correct to 4 to 5 decimal places we have to consider million of terms. Mādhava knows the fact of end correction used in infinite series. Using the end correction he has given accurate value of π correct to 11 decimal places. The method used by Mādhava is known as *antya-saṃskāra*. The name itself indicates that the series is terminated after considering only some number of terms from the beginning.

6. **Conclusions**

1. Indian mathematicians produced a number of works with idea of calculus. The formula for sum of cubes of first n natural numbers was given by Āryabhata in 500A.D. in order to find the volume of a cube. This was an important step in development of integral calculus.

2. Indian mathematician and astronomer Āryabhata used the notation of infinitesimals and expressed an astronomical problem in the form of basic differential equation.

3. This equation led Bhāskarācārya (12^{th} century) to develop the concept of derivative by stating
   $d(\sin\theta) = (\cos\theta)\, d\theta$
   representing the infinitesimal change in θ and then stated an early form of what is now known as Rolle's mean valus theorem.

4. In 15^{th} century, the early version of mean value theorem was first described by Parameśvara (1360 – 1460) in his commentaries on the work of Bhāskarācārya.

5. In 14^{th} century, a pioneer of Kerala School of mathematics and astronomy – Mādhava of Saṅgamagrāma described a special case of Taylor's series and now these series are known as Mādhava - Taylor's series, similarly Mādhava – Maclaurin's series etc.

6. Mathematical analysis has its roots in the work done by Mādhava of Saṅgamagrāma in 14^{th} century. The power series of sine and cosine are now known as Mādhava – Newton series. A text *Yukti-bhāṣa* by Jyeṣṭhadeva is considered to be the first text on calculus and it covers all these results.

7. Historically, there were four major steps in the development of today's concept of the derivative. The derivative was first **used**, then it was **discovered,** then it was **explored** and **developed**, and finally it **defined**. India mathematicians are using the concept of Infinitesimal calculus from the period of Āryabhata.

8. The important idea of "infinitesimal calculus" is "infinitely large" and "Infinitely small". **So when n increases Sama-ghāta-saṅkalita is the integration and Vārasaṅkalita is integration of integration.**

9. Mādhava (Founder of Kerala School Mathematics 1340 – 1425) found the value of π using by the method of calculating circumference without finding square roots is based on concept of infinite series and the value of π in terms of infinite series. This series converges slowly.

10. The infinite series is credited to Mādhava but quoted by Śaṅkara Vāriyar in his commentary *Yukti-dīpikā* and *Kriyā-kramakarī*. **The method of calculating circumference without finding square roots is the one of the contributions of Mādhava based on concept of infinite series. Also it covers the concept of Integration.**

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